AIAA Journal
Vol. 39, No. 5, May 2001

Theory for Conical Membrane Wings of High Aspect Ratio

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The inviscid aerodynamics of high-aspect-ratio wings having conical surfaces that are inextensible membranes capable of carrying compressive stresses and supported along two straight generators are investigated. For a given shape, the aerodynamics of the wing are found using lifting-line theory and thin-airfoil theory. The observation that the line of action of the membrane tension along each supporting generator is the same gives rise to a global constraint between the membrane shape, the pitching moment, and the root bending moment. This constraint enables an approximate solution to be found for the equilibrium membrane shape without needing to consider the full equations of equilibrium for the membrane. The functional form of the relationships between the wing shape and its force and moment coefficients is found to be analogous to those for the corresponding two-dimensional problem. The wing shapes considered in detail are the wing shape needed for elliptic loading, a triangular planform and an optimum wing at off-design conditions.

Nomenclature

b, c_0, Λ	=	semispan, centerline chord, and aspect ratio
C_L, C_{Di}, C_T	=	force coefficients for overall lift, induced drag, and tension (for two-dimensional sail)
$C_l(\eta), C_m(\eta)$	=	coefficients for sectional lift and leading-edge pitching moment
$C_{\mathrm{MP}}, C_{\mathrm{MB}}$	=	coefficients for overall pitching moment and root bending moment
C_{MR}	=	coefficient for the moment about the z axis, extending the membrane, $M_R/\frac{1}{2}\rho V^2 b^2 c_0$
$f(\eta), A_1, A_2$	=	wing shape function and two parameters that define it
$S(y) = c_0 c(y)$	=	chord shapes
(x, y, z) and (r, θ, z)	=	Cartesian and cylindrical coordinate systems
$\alpha, \alpha_i(\eta), \alpha_0(\eta)$	=	geometric, induced, and zero-lift angles of attack
β	=	leading-edge sweep angle
ε, Θ	=	excess length of two-dimensional sail and excess angle for three-dimensional sail
η	=	spanwise dimensions, y/b and $\cos(\phi)$

Introduction

THE analysis of potential flow around flexible membrane wings in two dimensions was first carried out by Von Voelk,[1] Nielsen,[2] and Thwaites,[3] each of whom assumed small incidence and camber. In each case the results of thin-airfoil theory were combined with the equation of equilibrium for the membrane, showing that the coefficients of lift and sail tension, C_L and C_T, were related to the angle of attack α and the excess length ε of the sail by functions of the form

$$C_T = f(\alpha/\sqrt{\varepsilon}), \qquad C_L/\sqrt{\varepsilon} = f(\alpha/\sqrt{\varepsilon})$$

Subsequent authors extended the linear potential theory to account for large displacements, porosity, double surface, and interacting sails; a review of this work may be found in Ref. 4. More recently, laminar and turbulent viscous effects have been added.[5–7] The study of thin, rigid, two-dimensional sections is important because the flow over three-dimensional wings is predominantly chordwise, and so advances in flow models for two dimensions can be carried over to three-dimensional problems. However, there is little more to be learned from the behavior of flexible thin surfaces because the boundary conditions for the two cases are quite different, in two dimensions the wing is supported at both the leading and trailing edges, whereas in three dimensions there is almost never any trailing-edge support. The structural behavior of the two kinds of wings is, therefore, completely different.

However, the theory of three-dimensional membrane wings is much less well developed. This is partly because analytical approaches have been overtaken by numerical solutions to the relevant equations, most often by combining a panel method with a finite element method.[8–12] This approach has the advantage of being able (in principle) to cope with the more subtle aspects of membrane wing design, such as anisotropy and wrinkling of the fabric or bending of the wing supports. Its primary disadvantage is that every solution is unique, so that it is difficult to discern any underlying relationship between the dependent and independent variables.

However, the main obstacle to analytical solutions for three-dimensional wings is simply that the general form of the coupled aerodynamic and structural equations is sufficiently complex as to defeat attempts to find general solutions. Nevertheless, some progress with semi-analytical methods has been made for special cases. By choosing appropriate boundary conditions, for example by restraining the trailing edge with a cable, a number of authors have assumed the membrane stresses to run only in the chordwise direction.[13–16] Holla et al.[17] treated elastic rectangular membranes fully restrained at the edges and with high pretension, whereas Sugimoto[18] obtained solutions for a fully restrained elastic circular wing by assuming that the tension remains equal in the spanwise and chordwise directions. However, perhaps the most significant advance was that made by Letcher,[19] who pointed out that the structural analysis is greatly simplified if the membrane has a developable surface. Letcher gave a few simple examples for cylindrical and conical shapes, and the idea was picked up and developed further by Sugimoto[20] for wings of cylindrical shape and restrained only at the leading edge and at the center of the trailing edge. This was the first example of a semi-analytical solution of a three-dimensional membrane wing with realistic boundary conditions.

However, there is still no corresponding method for analyzing membrane wing shapes of the most common kind, those that are restrained along two or more straight radial generators. Examples of this kind of wing include Rogallo wings (kites and simple hang gliders) and the mainsails of yachts. This paper, therefore, develops just such a theory, under the additional constraint that the entire surface is developable (conical). The genesis of the method lies in the paper by Letcher,[19] combined with the discovery by Jackson[21] that the structural behavior of a two-dimensional membrane can be adequately modeled by considering the overall equilibrium of forces on

Received 16 January 2000; revision received 19 July 2000; accepted for publication 21 July 2000.

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the wing, without needing to solve the membrane equation everywhere. A similar idea is employed here for three-dimensional wings of conical shape, together with lifting-line and thin-airfoil theory, to determine the aerodynamic behavior. The usual assumptions for these theories therefore apply, namely, that the wing has small camber and incidence and a high aspect ratio. Note that the membrane is supposed to be able to carry compressive stresses where necessary: The surface is, therefore, a membrane in the sense of shell theory, able to resist wrinkling.

Governing Equations

Wing Shape

The shapes of interest are those formed by an inextensible, flexible membrane that is constrained to a conical surface by its two straight leading edges with sweep angle β and a chordline at the center that is also a straight line. The surface is generated by straight lines radiating from the origin at the intersection of these constrained edges, as shown in Fig. 1a. With these edges lying in the x–y plane, the surface is then determined by the function $f(\theta)$:

$$z = rf(\theta) \tag{1}$$

where the angle θ is defined by Fig. 1b and $f(\beta) = 0$. [Note that the following analysis does not presume that $f(0) = 0$, to allow a second kind of wing that is restrained only by its leading edges and not along its centerline.] The sectional slope at fixed y is then given by

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \cos\theta f(\theta) - \sin\theta f'(\theta) \tag{2}$$

The planform has a root chord c_0 in the x direction and extends to a semispan b in the y direction and is described by the chordline function

$$S(y) = c_0 c(\eta)$$

where $\eta = y/b$. This planform and the function f are all that is needed to completely define the camber and twist at any wing section.

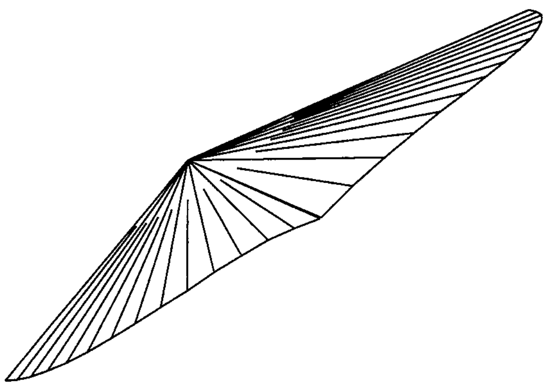

Fig. 1a General view of a conical membrane wing.

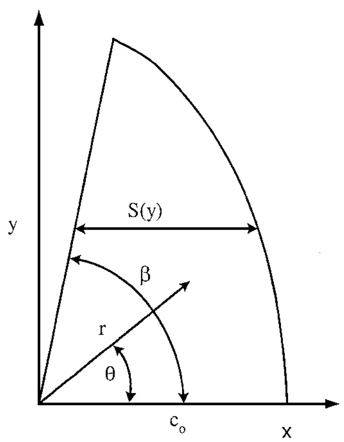

Fig. 1b Axes and dimensions for the wing planform.

Because the membrane is an inextensible conic, the total arc length at any radius must increase linearly with the radius. That is,

$$\int_0^\beta \sqrt{(r\,\mathrm{d}\theta)^2 + (\mathrm{d}z)^2} = r(\Theta + \beta)$$

which for small slopes reduces to

$$\frac{1}{2}\int_0^\beta (f')^2\,\mathrm{d}\theta = \Theta \tag{3}$$

We call Θ the excess angle because it is the increment that must be added to β to flatten the membrane to a planar surface. It is this angle that determines the magnitude of the curvature in the membrane.

Classical Aerodynamics for Thin Wings

For a given planform $c(\eta)$ and shape function $f(\theta)$, the sectional shapes are completely prescribed, and so, in principle, it is then possible to find the distribution of pressure jump over a section at any given effective angle of attack. Classical thin-airfoil theory gives the sectional coefficients for lift and leading-edge moment as (see for example, Ref. 22)

$$C_l = 2\pi(\alpha - \alpha_i - \alpha_0), \qquad C_m = (C_l/4) + C_{m0} \tag{4a}$$

where the zero-lift angle and moment are given by

$$\alpha_0(\eta) = \frac{1}{\pi}\int_0^\pi \frac{\mathrm{d}z}{\mathrm{d}x}(1 - \cos\hat{\vartheta})\,\mathrm{d}\hat{\vartheta}$$

$$C_{m0}(\eta) = \frac{1}{2}\int_0^\pi \frac{\mathrm{d}z}{\mathrm{d}x}(1 + \cos\hat{\theta} - 2\cos^2\hat{\vartheta})\,\mathrm{d}\hat{\vartheta} \tag{4b}$$

$$\frac{\hat{x}}{S(\eta)} = \frac{1}{2}(1 - \cos\hat{\vartheta}) = \frac{b}{c_0}\frac{\eta}{c(\eta)}\left(\frac{1}{\tan\theta} - \frac{1}{\tan\beta}\right) \tag{4c}$$

If the shape function f is known, the last equation may be used with Eq. (2) to find the sectional slope needed to evaluate the given integrals.

The induced downwash angle $\alpha_i(\eta)$ is a function of the three-dimensional aerodynamics. By the making of the transformation $\eta = \cos\phi$ and expressing the sectional loading as a Fourier series, lifting-line theory[22] enables the downwash to be written as a second, related series. Equation (4a) then becomes

$$C_l c = \frac{2b}{c_0}\sum_{n=1}^{\infty} B_n \sin n\phi = 2\pi\left[\alpha - \sum_{n=1}^{\infty}\frac{B_n n \sin n\phi}{4\sin\phi} - \alpha_0(\phi)\right]c(\phi) \tag{5}$$

with the overall lift and drag coefficients being given by

$$C_L = \frac{\pi}{4}\Lambda B_1, \qquad C_{Di} = \frac{\pi}{16}\Lambda\sum_{n=1}^{\infty} nB_n^2 \tag{6}$$

where Λ is the aspect ratio of the wing. For a given wing shape, the functions $f(\theta)$ and $c(\eta)$ are known, and so Eqs. (4b) and (4c) determine the zero-lift angle $\alpha_0(\eta)$. Equation (5) may then be used to find the B_n coefficients in the usual way, by truncating the series to N terms and satisfying the equation at N values of η or ϕ.

This section contains all of the equations needed to find the aerodynamic loading of a given wing shape. However, although there are an infinite number of possible wing shapes $f(\theta)$ that comply with the edge boundary conditions for a membrane wing of given planform $c(\eta)$, only one can be the actual shape taken up by a membrane under its corresponding aerodynamic loading. It is, therefore, necessary to find the additional constraints on possible wing shapes that follow from the structural equilibrium of the membrane.

Moment Equilibrium of the Sail Structure

Because in this model there are no tangential stresses on the membrane, the tension normal to any radial cut in the membrane from the origin to the trailing edge must carry a moment about the z axis, which is independent of the angle at which the cut is made. This moment is related to the tension at the cut $T(r,\theta)$ by the following equation. This also serves to define the moment coefficient C_{MR} for the restraining moment that must be provided to keep the membrane extended:

$$M_R = \int_0^{R(\theta)} T(r,\theta) r \,\mathrm{d}r = \frac{1}{2}\rho V^2 b^2 c_0 C_{\mathrm{MR}} \tag{7}$$

The membrane attaches at an angle $f'(0)$ along the centerline and along the leading edges at an angle $-f'(\beta)$. For one-half of the wing, the root bending moment about the centerline M_B and the pitching moment about the nose M_P can then be given in terms of the membrane tension or, using the aforementioned result, in terms of the restraining moment M_R:

$$M_B = -\int_0^{R(\beta)} rT(r,\theta)\sin\beta f'(\beta)\,\mathrm{d}r = -M_R[\sin\beta f'(\beta) + f(0)] \tag{8a}$$

$$M_P = -\int_0^{R(\beta)} rT(r,\theta)\cos\beta f'(\beta)\,\mathrm{d}r + \int_0^{R(0)} rT(r,\theta) f'(0)\,\mathrm{d}r$$
$$= M_R[f'(0) - \cos\beta f'(\beta)] \tag{8b}$$

The term $f(0)$ in the expression for the bending moment allows for the contribution that occurs when the membrane at the centerline is not attached to the x axis, a result that will be needed for the one-lobe wing studied later.

Alternative expressions for both moments may also be found from the usual coefficients for the sectional lift and leading-edge moments acting at a two-dimensional section of the membrane at span y:

$$M_B = \frac{1}{2}\rho V^2 \int_0^b yS(y)C_l(y)\,\mathrm{d}y$$
$$M_P = \frac{1}{2}\rho V^2 \int_0^b \left[S^2(y)C_m(y) + \frac{y}{\tan\beta}S(y)C_l(y)\right]\mathrm{d}y \tag{9}$$

By the use of the definitions

$$I_1 = \int_0^1 \eta c(\eta)C_l(\eta)\,\mathrm{d}\eta, \qquad I_2 = \int_0^1 c^2(\eta)C_m(\eta)\,\mathrm{d}\eta$$

and by the definition of coefficients for the moments in the same way as for M_R, these equations may be rearranged into the form

$$C_{\mathrm{MB}} = -C_{\mathrm{MR}}[\sin\beta f'(\beta) + f(0)] = I_1$$
$$C_{\mathrm{MP}} = (C_{\mathrm{MB}}/\tan\beta) + (c_0/b)I_2 \tag{10a}$$
$$\kappa = \frac{c_0}{b}\frac{I_2}{I_1} = -\frac{f'(0) + \cot\beta f(0)}{f(0) + \sin\beta f'(\beta)} \tag{10b}$$

These are the required global constraints arising from the membrane equilibrium. The first two link the coefficient for the membrane restraining moment C_{MR} to the aerodynamic moments on the sail. The third provides a relationship between the distributions of the sectional lift and moment coefficients and the membrane shape. With these expressions, it is then possible to find an approximate solution for the equilibrium membrane shape, without needing to solve the full equation of equilibrium for the membrane at every point (as discussed hereafter).

General Form of the Solution

To proceed further, it is necessary to assume a functional form for a conical shape. Jackson[21] obtained a good approximation to the exact case in two dimensions by using a cubic, and so here we shall assume

$$f(\theta) = A_1(\theta/\beta)(1-\theta/\beta)(1 + A_2 - 2A_2\theta/\beta)$$

which has $f(\beta) = f(0) = 0$. This particular form is chosen so that its value at $\beta/2$ is determined only by A_1 and so that the ratio of slopes at the two ends is determined only by A_2. For a wing of known planform, the ratios α_0/α and C_l/α and the moment integrals I_1/α and I_2/α then become functions of the independent variables A_1/α, A_2, β, and c_0/b. The first of these may be eliminated using Eq. (3), which becomes

$$\frac{A_1}{\alpha} = \pm\frac{\sqrt{\Theta}}{\alpha}\sqrt{\frac{30\beta}{5+3A_2^2}} \tag{11}$$

and Eq. (10b) may be then used to fix the value of A_2. It follows that, if the sweep angle and span/chord ratio are specified, the geometry and aerodynamics of a developable membrane wing then depend only on the excess angle scaled as shown. In particular,

$$C_L/\sqrt{\Theta} = g_1(\alpha/\sqrt{\Theta}), \qquad C_M = g_2(\alpha/\sqrt{\Theta})$$

These useful results are a close analog of the corresponding scaling for a two-dimensional membrane wing,[2] where the results are the same except that the excess angle is replaced by the excess length in the membrane arc and the restraining moment by the membrane tension.

Note that A_1 has two values for each value of A_2. This means that it is also possible to find an equilibrium shape for wings that are deflected downward rather than upward [the negative value in Eq. (11)]. The pressure loads are then carried by compression in the membrane, which, if it were unable to sustain such loads, would snap through to another solution.

Finally, a useful measure of the degree to which the trailing edge is deflected is given by the twist at the wing tip α_t, which is easily shown to be given by

$$\alpha_t/\alpha = (\sin\beta/\beta)(A_1/\alpha)(1 - A_2)$$

Wing Shapes for Elliptic Loading

The first problem of interest is to find the wing shape that results in the least drag for a given lift. It is well known that the required loading is elliptic; that is, $B_n = 0$ for $n > 1$ [as is obvious from Eq. (6)]. According to Eq. (5), the wing chord required must then satisfy

$$C_l(\eta)c(\eta) = 2B_1(b/c_0)\sqrt{1-\eta^2} = 2\pi(\alpha - \alpha_0(\eta) - B_1/4)c(\eta)$$

and so, by the use of $c(0) = 1$,

$$\alpha_i = \frac{B_1}{4} = \frac{\alpha - \alpha_0(0)}{1 + (4/\pi)(b/c_0)} \tag{12a}$$

$$\frac{c(\eta)}{\sqrt{1-\eta^2}} = \left[1 - \frac{\alpha_0 - \alpha_0(0)}{\alpha - \alpha_0(0)}\left(1 + \frac{\pi}{4}\frac{c_0}{b}\right)\right]^{-1} \tag{12b}$$

Thus, the departure of the wing from an elliptic profile depends on the magnitude of α_0/α. If this equation is used to eliminate $c(\eta)$ from Eq. 4c, then the first of Eqs. (4b) becomes a rather complex integral equation for α_0/α at any spanwise position, if A_1/α and A_2 are known. Once solved, the corresponding planform shape follows from Eq. (12b).

To find A_1/α and A_2, Eq. (11) is first used to find A_1/α in terms of A_2 and the excess angle. Then, using the shape function $f(\theta)$ as defined earlier, the expression on the right side of Eq. (10b) may be evaluated as an explicit function of β and A_2, with the result

$$A_2 = -\frac{1 - \kappa\sin\beta}{1 + \kappa\sin\beta} \tag{13}$$

This couples the necessary wing shape for elliptic loading back to the aerodynamics only through the ratio I_2/I_1 (or κ), for which the necessary integrands are determined by the aerodynamic set in Eqs. (4). This completes the full set of equations needed to find the wing shape.

Before proceeding to the general solution, it is helpful to consider the limiting case of a vanishing excess angle, when the membrane will become nearly flat, so that α_0 and C_{m0} will also vanish. The necessary shape for elliptic loading will itself become elliptic, and it may then be shown that $I_1/I_2 \to 2.0$, so that the limiting value for the wing shape parameter A_2/A_1 is, from Eq. (13),

$$A_{2e} = -\frac{1 - (c_0/2b)\sin\beta}{1 + (c_0/2b)\sin\beta}$$

Because the loading is elliptic, I_1 can be evaluated analytically, and Eqs. (10) then give the limiting values of the expressions for the moments:

$$C_{\mathrm{MR}}\frac{\sqrt{\Theta}}{\alpha} = \frac{2\pi}{3[1 + (\pi c_0/4b)]}\frac{\sqrt{\beta/30}}{\sin\beta}\frac{\sqrt{5 + 3A_{2e}^2}}{1 - A_{2e}}$$

$$\frac{C_{\mathrm{MP}}}{\alpha} = \frac{2\pi}{3[1 + (\pi c_0/4b)]}\left(\frac{1}{\tan\beta} + \frac{c_0}{2b}\right)$$

The general case lends itself to an iterative solution using Eq. (13) as a global constraint on the entire wing. For a given β and c_0/b, we fix the value of A_1/α and guess the corresponding value of A_2. This allows the wing shape to be found as described, along with its sectional lift and moment coefficients. The latter are then used to evaluate I_1/I_2, which in turn allows a new value for A_2 to be found from Eq. (13). Once the solution converges, the corresponding excess angle and moments may be found from Eqs. (10a) and (11).

Figure 2 and Table 1 show the results of calculations for $\beta = 90$ deg and $c_0/b = 0.4$, obtained using MATLAB®, with 41 points cosine spaced along the span to evaluate the moment integrals I_1 and I_2. Figure 2 is an isometric view of the wing planform needed for elliptic loading when $\sqrt{\Theta}/\alpha = 0.2$ and its equilibrium shape when flying. As expected, the chord near the tips is greater than that of an ellipse because the wing twist means that increased wing area is needed there to maintain the elliptic loading. Table 1 shows how the various parameters of the solution vary with the excess angle in the membrane; note that neither $A_1/\sqrt{\Theta}$ nor the scaled restraining moment $C_{\mathrm{MR}}\sqrt{\Theta}/\alpha$ vary much with the scaled excess angle. At higher values of $\sqrt{\Theta}/\alpha$, the planform shape becomes quite distorted, with the outboard part of the wing having a chord significantly greater than the root chord. With other values of sweep and aspect ratio, the shapes of these curves were found to be very similar when scaled by their values at zero slackness, and so results for this limit, derived earlier, when combined with Table 1, serve to predict membrane behavior over the entire range of independent parameters. However, it is important to remember that although this method may be used to find the optimum planform for a wing of given c_0/b, β, and Θ at a particular angle of attack, the wing will have elliptic loading only at that angle of attack.

Table 1 Shape parameters needed for elliptic loading, and the resulting moment coefficients, for various values of the excess angle ratio

$\sqrt{\Theta}/\alpha$	$A_1/\sqrt{\Theta}$	A_2	$C_{\mathrm{MR}}\sqrt{\Theta}/\alpha$	C_{MP}/α
0	2.73	−0.667	0.551	0.319
0.1	2.73	−0.631	0.556	0.360
0.2	2.79	−0.590	0.563	0.411
0.3	2.83	−0.542	0.573	0.472

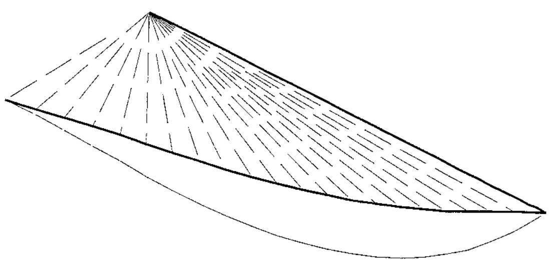

Fig. 2 View of the planform needed for elliptic loading and the resulting equilibrium shape when $c_0/b = 0.4$ and $\alpha/\sqrt{\Theta} = 5.0$.

A developable wing may also be formed by a membrane supported only at its leading edges, provided that $\beta < \pi/2$ so that a pitching moment may be sustained. Such a single-lobe wing will have a shape f that is even in θ, so that an appropriate shape function might be

$$f(\theta) = A_1(1 - \theta^2/\beta^2)\left[1 + A_2/5 - A_2(\theta^2/\beta^2)\right] \tag{14}$$

which results in an expression similar to Eq. (11). Wing shapes and other data may be found as before, but no results are given here because they are do not show any strikingly different features to those of the two-lobed wing presented earlier.

Wings of Known Planform

The second problem of interest is to find the flying shape and aerodynamic behavior for a wing of known planform and excess angle. For an assumed starting value of κ, Eqs. (11) and (13) fix the remaining wing shape parameters A_1/α and A_2, so that Eqs. (4b) and (4c) may be used to evaluate the distribution of α_0/α and C_{m0}/α along the wing. The terms in the lifting-line equation, Eq. (5), may then be evaluated at collocation points along the wing, allowing the resulting set to be solved for the corresponding number of coefficients B_n. These determine the distribution of the sectional lift and moment along the wing, so that the integrals I_1 and I_2 may be evaluated and used in Eq. (10b) to update the value of the starting parameter κ. The entire procedure is then repeated until it converges. Here, 41 collocation points were used, iterating until κ was accurate to 0.01%.

The first results of this process are shown in Fig. 3 for a wing with an elliptic chord function and $c_0/b = 0.4$, for a range of values of $\alpha/\sqrt{\Theta}$. When this parameter is large, the excess angle is small compared with the angle of attack, and so the effect of camber becomes negligible and the wing takes on elliptic loading (with the various parameters tending to their limiting values as described earlier). The sectional lift is then evenly distributed, but because the wing tip is less well supported than the inboard part, it tends to twist more. This can be seen in the top part of Fig. 3, which shows the deflected shape of the trailing edge ($z/\sqrt{\Theta}$). As the angle of attack decreases (for a fixed excess angle), the tip then tends to unload first and the inboard wing becomes relatively more loaded, with the distribution of twist then altering accordingly. As shown in Fig. 3, for smaller angles of attack the loading at the outboard part of the wing changes sign, whereas the wing is inflated normally. However, eventually this process must cause the wing to collapse.

The remaining parameters for this wing are shown in Figs. 4a and 4b, for both the positive and negative roots for A_1. At larger angles of attack, A_2 is negative, but for a wing in tension (deflected

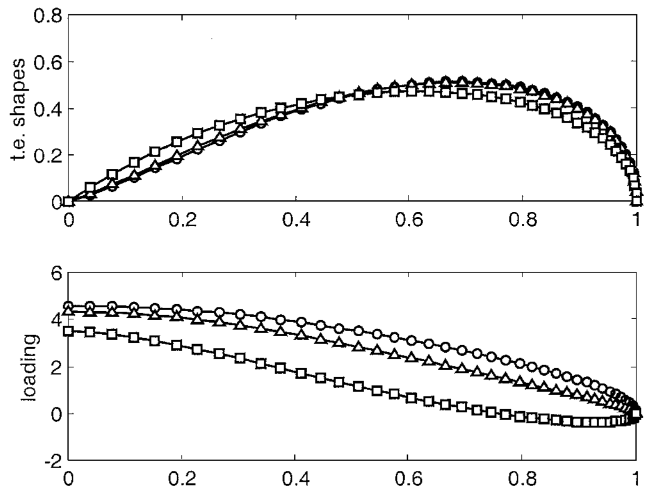

Fig. 3 Trailing-edge deflection (scaled by $\sqrt{\Theta}$) and loading distribution for a wing of elliptic planform, $c_0/b = 0.4$ and no sweep ($\beta = 90$ deg), for various values of $\alpha/\sqrt{\Theta}$: ○, $\alpha/\sqrt{\Theta} = 5.0$; △, 2.0; and □, 1.25.

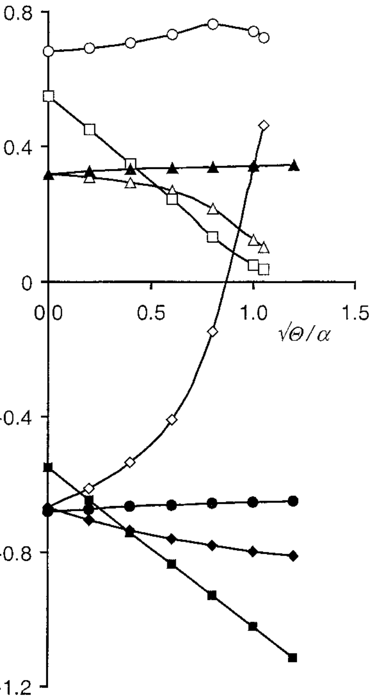

Fig. 4a Shape and moment parameters for the wing of Fig. 3, as a function of $\alpha/\sqrt{\Theta}$ (note that A_1 is shown scaled by one-quarter and that data for negative A_1 are shown in black symbols): □, $C_{MR}\sqrt{\Theta}/\alpha$; △, C_{MP}/α; ○, $A_1/4\sqrt{\Theta}$; and ◇, A_2.

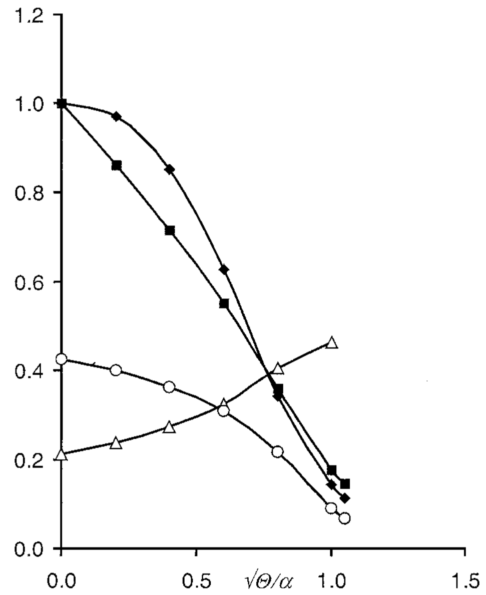

Fig. 4b Aerodynamic parameters for the wing of Fig. 3; ratio of lift to that of an elliptic wing, elliptic drag to actual drag, and chordwise and spanwise centers of effort, as functions of $\alpha/\sqrt{\Theta}$: ■, lift efficiency; ♦, drag efficiency; △, C_x/c_0; and ○, C_y/b.

upward) as α decreases, A_2 changes sign and the moments decrease until they vanish, at which point the wing collapses. The data for negative values of A_1 are hardly of practical relevance but are shown for completeness. As the excess length increases, the downward deflection of the wing increases, but with little change in wing shape (both A_1 and A_2 are fairly constant), and so consequently the pitching moment does not change either. The aerodynamic behavior of the wing is shown in Fig. 4b (for the wing in tension only). Here the lift is given as a ratio to the value for an elliptically loaded wing with the same aspect ratio [using Eqs. (6) and (12a), this is $\pi\alpha\Lambda/(1+4b/\pi c_0)$, where $\Lambda = 6.37$], the drag is given as ratio of an elliptically loaded wing to the actual drag (the so-called efficiency factor), and the centers of effort are given as fractions of the centerline chord and span. As expected, the efficiency factors approach unity at high angles of attack, when the loading due to incidence dominates loading due to camber and the wing approaches elliptic loading. At the smaller angles, the wing rapidly becomes less efficient.

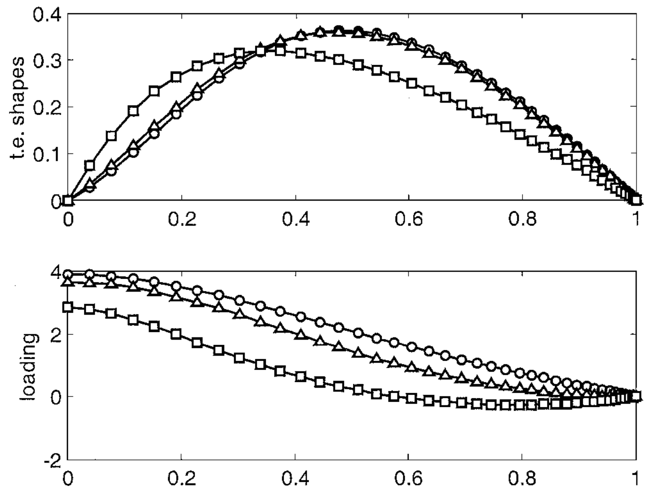

Fig. 5a Trailing-edge deflection and loading distribution for a wing with a linear chord distribution, $c_0/b = 0.4$, and straight trailing edge, for various $\alpha/\sqrt{\Theta}$ (5.0, 2.0, and 1.25, as in Fig. 3).

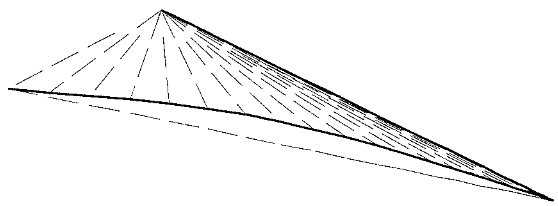

Fig. 5b Flying shape for the wing of Fig. 5a, at $\alpha/\sqrt{\Theta} = 5.0$.

As the next example, Fig. 5a shows the trailing-edge shape and spanwise loading for a wing with a planform that is triangular in shape, with $c_0/b = 0.4$ and with a straight trailing edge (that is, $\beta = a\tan(1/0.4) = 62.8$ deg), whereas Fig. 5b shows a view of the flying shape when $\sqrt{\Theta}/\alpha = 0.2$. It can be seen that, compared to the elliptic wing, the loading has moved inboard, following the redistribution of area. Because the wing tip is now better supported, the twist is less there, and so the largest edge deflections now occur nearer to the center of the span. Results for wings of other simple planform show that they exhibit similar behavior. The only feature of note is that, as one might expect, decreasing β also causes the membrane restraining moment to decrease.

The final problem of interest is the behavior of a wing designed for elliptic loading (using the results of the preceding section) when it is not at its design angle of attack. The particular wing chosen has $c_0/b = 0.4$, $\beta = 90$ deg, and elliptic loading when $\sqrt{\Theta}/\alpha = 0.2$, for which Fig. 6 shows the primary results for a range of angles of attack. In general, the wing behaves in the same way as the elliptic wing described earlier, although the drag efficiency now holds up over a wider range of angles around the design point.

The whole analysis may be simplified in the case of large aspect ratio, when Eqs. (10b) and (13) show that $A_2 \to -1$, so that Eq. (11) gives $A_1/\sqrt{\Theta} \to \sqrt{(15\beta/4)}$. This fixes the shape completely and, because the induced angle also becomes negligible, all of the coefficients then reduce to one term dependent on α and another on $\sqrt{\Theta}$. For an elliptic planform, for example, we find

$$C_L = 2\pi(\alpha - 3.01\sqrt{\Theta})$$

$$C_{MR}\sqrt{\Theta} = (\pi/3\sin\beta)\sqrt{(4\beta/15)}(\alpha - 3.09\sqrt{\Theta})$$

with Eqs. (10a) giving all of the remaining moment coefficients. With $\beta = \pi/2$, this becomes $C_{MR}\sqrt{\Theta} = 0.677(\alpha - 3.09\sqrt{\Theta})$, so that, by comparison with the results in Fig. 4, we may conclude

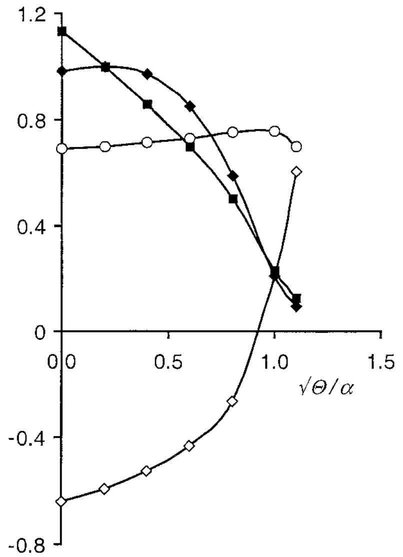

Fig. 6 Shape and aerodynamic parameters for the membrane wing having the planform that generates elliptic loading at $\alpha/\sqrt{\Theta} = 5.0$, with $c_0/b = 0.4$, $\Lambda = 5.30$, $\beta = 90$ deg (note that A_1 is shown scaled by one-quarter): ○, $A_1/4\sqrt{\Theta}$; ◇, A_2; ■, lift efficiency; and ♦, drag efficiency.

that increasing the aspect ratio has only a small effect on C_L/α or $C_{MR}\sqrt{\Theta}/\alpha$ at large values of $\alpha/\sqrt{\Theta}$, but that it tends to reduce the angle at which the wing collapses, to reach a limiting value of around $\alpha = 3\sqrt{\Theta}$.

Conclusions

A solution for a fully three-dimensional conical membrane wing has been found using thin-airfoil and lifting-line theory. Besides the usual approximations inherent in these theories, the solution is approximate in the sense that the equilibrium equations for the membrane have been satisfied globally, rather than at every point. This approach results in a relationship between the membrane shape and the resulting pitching and bending moments that allows the angle at which the membrane attaches to its supports to be found, and these in turn are all that is needed to find a good approximation to the equilibrium shape.

All of the resulting force and moment coefficients have been shown to be functions only of the angle of attack scaled by the square root of the excess angle in the membrane (a close analogy to the result for two-dimensional wings). A process for obtaining the planform shape needed to obtain elliptic loading at a particular design point has been described and illustrated. Three examples are given for the shape and coefficients of wings with elliptic and triangular planforms and for an optimum wing that is operating off its design point.

This theory also makes possible some important extensions for this kind of wing (all of which are in progress), including the distribution of stress within the membrane, the aerodynamic stability of conical wings, and flapping membrane wings.

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